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The numerical solution of boundary-value problems for an elastic body with an elliptic hole and linear cracks

N. Zirakashvili

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Abstract Using the boundary-element method which is a combination of a fictitious load and a displacement discontinuity, numerical solutions are obtained for two-dimensional (plane deformation) boundary-value problems for the elastic equilibrium of infinite and finite homogeneous isotropic bodies having elliptic holes with cracks and cuts of finite length. Using the method of separation of variables, the boundary-value problem is solved in the case of an infinite domain containing an elliptic hole with a linear cut on whose contour the symmetry conditions are fulfilled.

Keywords Boundary element · Displacement discontinuity · Fictitious load · Separation of variables

1 Introduction

In the literature on cracks emanating from the surfaces of elliptic holes the values of critical loads causing the development of cracks are calculated. In [1–3], problems of ultimate equilibrium are solved in closed form for a brittle plate weakened by an elliptic hole with one or two small linear cracks located at the ends of the hole. In [4] stress intensity factors are considered for cracks emanating from elliptic holes in finite or infinite plates.

The present paper and the author's earlier work [5] deal with the question whether cracks can be helpful in strengthening structures. For example, when building underground structures, tunnels in particular, engineers intentionally make so-called technical openings in the tunnel walls in order to decrease the stress concentration and fortify the walls using various techniques.

In [5] we investigated how the number of cracks and their lengths influence the stress distribution in the tunnel walls, i.e., how the tangential stress concentration on the circular hole contour can be diminished by varying the number of cracks and their lengths.

In this paper we are concerned with the question how the tangential stress concentration can be diminished on the contour of an elliptic hole (except the crack ends) by varying the number of cracks and their lengths. Although the stress intensity factor near cracks is important, we do not consider it here since that will be the subject of a separate study.

N. Zirakashvili (✉)
I. Vekua Institute of Applied Mathematics, I. Javakhisvili Tbilisi State University, 2 University St., 0186 Tbilisi, Georgia
e-mail: natzira@yahoo.com

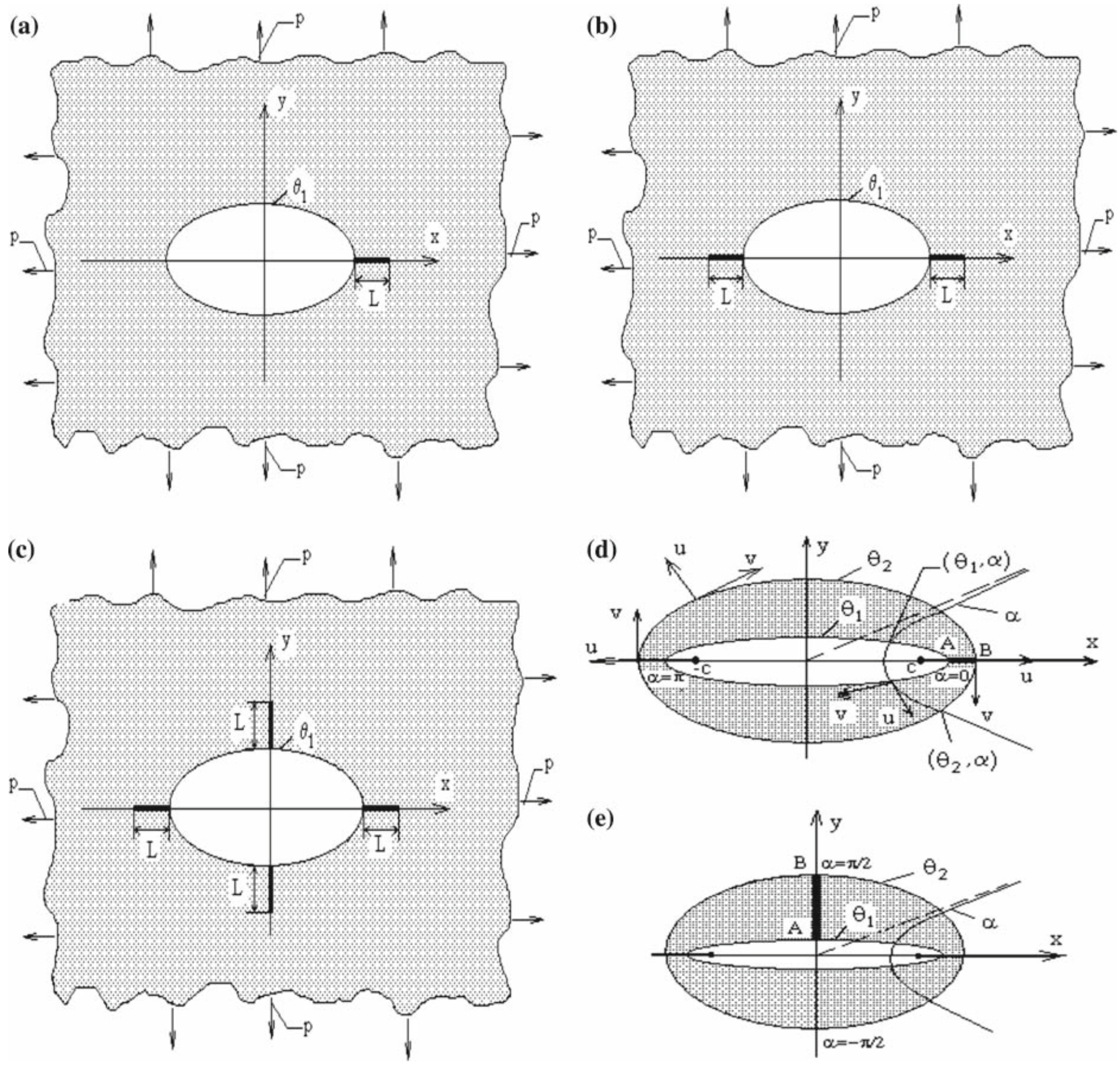

Fig. 1 The geometry and loading for the cases with (**a**) one crack, (**b**) two cracks, (**c**) four cracks, and (**d**) an elliptic ring with cut AB

In Sect. 2, using the boundary-element method (BEM) [6, Chaps. 4, 5], we solve the two-dimensional (plane-deformation) problem concerning the elastic equilibrium of an infinite homogeneous isotropic body having an elliptic hole with cracks of equal length L (Fig. 1, cases a, b, c). It is assumed that the body is free from internal stresses and that the biaxial tensile force is given at infinity. We obtain numerical results and present graphs for the cases with one, two and four cracks.

In Sect. 3, the elastic equilibrium of a finite body that occupies the domain $\Omega_1 = \{\theta_1 < \theta < \theta_2,\ 0 < \alpha < \pi\}$ (or $\Omega_2 = \{\theta_1 < \theta < \theta_2,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\}$) is defined in the elliptic coordinates θ, α. It is assumed that nonzero stresses are given for $\theta = \theta_1$, and zero stresses for $\theta = \theta_2$. The conditions of symmetry ($v = 0, \sigma_{\alpha\alpha} = 0 \Leftrightarrow v = 0,\ \frac{\partial u}{\partial \alpha} = 0$) and antisymmetry ($u = 0, \sigma_{\alpha\alpha} = 0 \Leftrightarrow u = 0,\ \frac{\partial v}{\partial \alpha} = 0$) [7] are given for $\alpha = 0$ and $\alpha = \pi$ (or for $\alpha = \frac{\pi}{2}$ and $\alpha = -\frac{\pi}{2}$), respectively. We easily observe that this problem coincides with the boundary-value problem concerning the elastic equilibrium of an elliptic ring with a cut on whose contours the symmetry conditions are fulfilled (Fig. 1, cases d, e).

2 Solution of the boundary-value problem for an infinite domain with an elliptic hole and cracks by the BEM

We write the equilibrium equations in terms of displacements as follows [8]:

$$\operatorname{grad}\left[(\lambda+2\mu)\operatorname{div}\overrightarrow{U}\right]-\mu\operatorname{rot}\operatorname{rot}\overrightarrow{U}=0,\quad \operatorname{div}(\mu\operatorname{rot}\overrightarrow{U})=0.$$

or

$$\operatorname{grad}D-\operatorname{rot}\overrightarrow{K}=0,\quad \operatorname{div}\overrightarrow{K}=0,\quad \operatorname{div}\overrightarrow{U}=\frac{1}{\lambda+2\mu}D,\quad \operatorname{rot}\overrightarrow{U}=\frac{1}{\mu}\overrightarrow{K}, \tag{1}$$

where $\lambda=\lambda(\theta,\alpha,\zeta)$, $\mu=\text{const}>0$ are classical elastic characteristics, $\overrightarrow{U}=\overrightarrow{U}(u,v,w)$ is the displacement vector; $\overrightarrow{K}(K_\theta,K_\alpha,K_\zeta)$; θ,α,ζ are the curvilinear coordinates.

Since here we deal with plane deformation, solutions of boundary-value problems are sought in the following form: $u=u(\theta,\alpha)$, $v=v(\theta,\alpha)$, $w=0$ in the domain bounded by the curves of an elliptic coordinate system θ,α $(0\le\theta<\infty,\ 0\le\alpha<2\pi$ [9, pp. 102–107]

$$\frac{x^2}{c^2\cosh^2\theta_0}+\frac{y^2}{c^2\sinh^2\theta_0}=1,\quad \theta_0=\text{const}\neq 0,$$

$$\frac{x^2}{c^2\cos^2\alpha_0}-\frac{y^2}{c^2\sin^2\alpha_0}=1,\quad \alpha_0=\text{const}\neq 0,\frac{\pi}{2},\pi,\frac{3}{2}\pi,$$

where $x,\ y$ are Cartesian coordinates, $x=c\cosh\theta\cos\alpha$, $y=c\sinh\theta\sin\alpha$ and c is a scale coefficient.

Projecting equalities (1) onto the tangential lines of the curves of the elliptic coordinate system, we obtain the following system of equilibrium equations:

$$\begin{aligned}&\frac{\partial D}{\partial\theta}-\frac{\partial B}{\partial\alpha}=0,\quad \frac{\partial\overline{u}}{\partial\theta}+\frac{\partial\overline{v}}{\partial\alpha}=\frac{æ-1}{æ\mu}h_0^2D,\\&\frac{\partial D}{\partial\alpha}+\frac{\partial B}{\partial\theta}=0,\quad \frac{\partial\overline{v}}{\partial\theta}-\frac{\partial\overline{u}}{\partial\alpha}=\frac{1}{\mu}h_0^2B,\end{aligned} \tag{2}$$

where $\overline{u}=\frac{2hu}{c^2}$, $\overline{v}=\frac{2hv}{c^2}$, $B=K_\zeta$, $h_0=\sqrt{\cosh(2\theta)-\cos(2\alpha)}$, $æ=4(1-\nu)$, $\mu=\frac{E}{2(1-\nu)}$; ν, E are known constants; $u,\ v$ are the components of the displacement vector in the system of elliptic coordinates θ,α; $h_\theta=h_\alpha=h=\frac{c}{\sqrt{2}}\sqrt{\cosh(2\theta)-\cos(2\alpha)}$ are the Lamé constants, $c=1$ is a scale coefficient; $\frac{æ-1}{æ\mu}h_0^2D$ is the divergence of the displacement vector, $\frac{1}{\mu}h_0^2B$ is displacement-vector rotor.

Now let us formulate the following boundary-value problem: in an infinite domain $\Omega=\{\theta_1<\theta<\infty,\ 0<\alpha<2\pi\}$ having an elliptic hole $\theta=\theta_1$ with cracks of equal length L, find a solution of the system of equilibrium equations (2) with respect to the unknowns D, B, $\overline{u}$, $\overline{v}$ using the boundary conditions

$$\text{when }\theta=\theta_1:\ \sigma_{\theta\theta}=0,\quad \sigma_{\theta\alpha}=0, \tag{3}$$

$$\text{when }\theta\to\infty:\ \sigma_{\theta\theta}=p,\quad \sigma_{\theta\alpha}=0, \tag{4}$$

$$\text{when }\alpha=0,\ 2\pi:\ \sigma_{\alpha\alpha}=0,\quad \sigma_{\theta\alpha}=0, \tag{5}$$

where $\theta_1<\theta<\theta_1+L$, and $\sigma_{\alpha\alpha},\sigma_{\theta\theta},\sigma_{\theta\alpha}$ are the stress-tensor components in the system of elliptic coordinates.

For the numerical solution of the problem we use a method based on a combination of a fictitious load [6, Chap. 4] and a displacement discontinuity [6, Chap. 5], [10]. This boundary-value problem is a generalization of that for an infinite body with a circular hole and radial cracks [5].

To solve the external problem with given nonzero stresses at infinity, the boundary conditions must be formulated in terms of additional stresses [6, p. 73]. When doing so, we obtain the boundary conditions (3), (4) and (5) in the form

$$\text{when }\theta=\theta_1:\ \sigma_{\theta\theta}=-p,\quad \sigma_{\theta\alpha}=0, \tag{6}$$

$$\text{when }\alpha=0,\ 2\pi:\ \sigma_{\alpha\alpha}=-p,\quad \sigma_{\theta\alpha}=0 \tag{7}$$

for $\theta_1<\theta<\theta_1+L$.

If the boundary is divided into N segments (elements) of small length, then it can be assumed that constant normal $\sigma^i_{\theta\theta} = -p$ (or $\sigma^i_{\alpha\alpha} = -p$) and tangential $\sigma^i_{\theta\alpha} = 0$ stresses act on each i-th element over its entire length. Then the boundary conditions (6), (7) take the form

$$\text{when } \theta = \theta_1\text{: } \sigma^i_{\theta\theta} = -p, \quad \sigma^i_{\theta\alpha} = 0, \tag{8}$$

$$\text{when } \alpha = 0,\ 2\pi\text{: } \sigma^i_{\alpha\alpha} = -p, \quad \sigma^i_{\theta\alpha} = 0 \tag{9}$$

where again $\theta_1 < \theta < \theta_1 + L$.

For each boundary element we choose concentrated forces uniformly distributed throughout its length. For example, for the j-th element we assume a continuous distribution of tangential P^j_s and normal P^j_n stresses. Also, for the j-th element we have fictitious stresses P^j_s and P^j_n and also real stresses σ^j_s and σ^j_n induced by the stresses applied to all boundary elements.

Using the solution of Kelvin's problem for plane deformation [11, pp. 336–339] and the coordinate-transformation formulas [12, pp. 23–26] (in order to take into account the orientation segments), we can calculate the real stresses σ^i_s and σ^i_n at the midpoints of all the segments, $i = 1, \ldots, N_1$. Thus we obtain the formulas

$$\sigma^i_s \equiv \sigma^i_{\theta\alpha} = \sum_{j=1}^{N_1} \left(A^{ij}_{ss} P^j_s + A^{ij}_{sn} P^j_n \right), \qquad \sigma^i_n \equiv \sigma^i_{\theta\theta} = \sum_{j=1}^{N_1} \left(A^{ij}_{ns} P^j_s + A^{ij}_{nn} P^j_n \right), \tag{10}$$

where A^{ij}_{ss}, A^{ij}_{sn}, A^{ij}_{ns}, A^{ij}_{nn} are the boundary coefficients for the influence of stresses for the problem under consideration. For example, the coefficient A^{ij}_{ns} gives the real normal stress at the center of the i-th segment (σ^i_n) induced by the constant unit tangential load ($P^j_s = 1$) applied to the j-th segment.

Let us now consider the part of the domain which contains a crack having two contours. The fictitious-load method is not applicable for the solution of crack problems because the influence of elements of one contour cannot be distinguished from the influence of elements of the other contour. To solve problems of this kind we use another BEM which is called the displacement discontinuity method [6, Chap. 5]. The method is based on the analytic solution of the problem of an infinite plane where displacements undergo a constant discontinuity within the limits of a finite segment. The analytic solution of that problem was obtained by S.L Crouch [10].

If the crack is divided into N_2 segments (elements) of small length, then it can be assumed that the displacement discontinuity is constant throughout the lengths of each element. The influence of an individual elementary displacement discontinuity on displacements and stresses at an arbitrary point of an infinite rigid body can be defined by means of Crouch's analytic solution. For example, tangential and normal stresses at the i-th element center can be expressed in terms of displacement discontinuity components of the j-th element.

If we put an elementary displacement discontinuity on each of the segments along the crack, we obtain

$$\sigma^i_s \equiv \sigma^i_{\theta\alpha} = \sum_{j=N_1+1}^{N} \left(C^{ij}_{ss} D^j_s + C^{ij}_{sn} D^j_n \right), \quad \sigma^i_n \equiv \sigma^i_{\alpha\alpha} = \sum_{j=N_1+1}^{N} \left(C^{ij}_{ns} D^j_s + C^{ij}_{nn} D^j_n \right), \tag{11}$$

$i = N_1 + 1, \ldots, N = N_1 + N_2$,

where $C^{ij}_{ss}, C^{ij}_{sn}, C^{ij}_{ns}, C_{nn}$ are the boundary coefficients of stress influence. For example, the coefficient C^{ij}_{ns} gives the normal stress (σ^i_n) at the center of the i-th element, which is induced by the constant unit displacement discontinuity directed tangentially along the j-th element ($D^j_s = 1$).

For the boundary conditions to be satisfied on $\theta = \theta_1$, we use formulas (10) obtained by the fictitious-load method, while for cracks we use formulas (11) obtained by the displacement discontinuity method. Thus we come to the following system of $2N$ linear equations with $2N$ unknowns ($N = N_1 + N_2$)

$$
\begin{cases}
\sum_{j=1}^{N_1}\left(A_{ss}^{ij}P_s^j + A_{sn}^{ij}P_n^j\right) \\
\quad + \sum_{j=N_1+1}^{N}\left(C_{ss}^{ij}D_s^j + C_{sn}^{ij}D_n^j\right) = 0, \\
\\
\sum_{j=1}^{N_1}\left(A_{ns}^{ij}P_s^j + A_{nn}^{ij}P_n^j\right) \\
\quad + \sum_{j=N_1+1}^{N}\left(C_{ns}^{ij}D_s^j + C_{nn}^{ij}D_n^j\right) = -p,
\end{cases}
\quad i = 1, \ldots, N. \tag{12}
$$

The stresses P_s^j and P_n^j in these equations are fictitious values. We have introduced them as auxiliary unknowns and they have no physical meaning. However, linear combinations of fictitious loads (10) give us real tangential and normal stresses which are used to satisfy the boundary conditions, while the unknowns D_s^j and D_n^j represent discontinuous displacements.

After solving system (12) by any numerical method (here we have used the Gauss method), we can calculate the displacements and stresses at any point of the body except the points lying inside a circle with center in the middle of a boundary element and having a radius equal to the lengths of this element, certainly not counting the midpoint of the element [6, p. 79].

Using the MATLAB software we obtained numerical results and constructed graphs for the boundary-value problem (2), (3), (4), (5) (or (2), (8), (9)) for $\nu = 0.3$, $E = 7 \times 10^4\,(\mathrm{N/m^2})$, $N_1 = 360$, $N_2 = 20$, $p = 10\,(\mathrm{N/m^2})$, $\theta_1 = 1\,(\mathrm{m})$. In particular, we obtained numerical results for three cracks of lengths $L = 0.1$, 0.5 and 0.8 m.

In Fig. 2 we see the values of $\sigma_{\alpha\alpha}$ for $\theta = \theta_1$, $0 < \alpha < 2\pi$ and one crack along $\alpha = 0$ of lengths $L = 0.1, 0.5$ and 0.8. Figure 3 shows the values of $\sigma_{\alpha\alpha}$ for $\theta = \theta_1$, $0 < \alpha < 2\pi$ and two cracks along $\alpha = 0$ and $\alpha = \pi$ of lengths $L = 0.1, 0.5$ and 0.8. Figure 4 presents the values of $\sigma_{\alpha\alpha}$ for $\theta = \theta_1$, $0 < \alpha < 2\pi$ and four cracks along $\alpha = 0$, $\alpha = \frac{\pi}{2}$, $\alpha = \pi$ and $\alpha = \frac{3}{2}\pi$ of lengths $L = 0.1, 0.5$ and 0.8.

The above graphs enable us to draw the following conclusions. As the number of cracks and their lengths grow, the concentration of stresses $\sigma_{\alpha\alpha}$, strange as it might seem, decreases. Knowing this fact, engineers sometimes make so-called technical crevices in order to strengthen underground structures. Concerning the ends of the cracks, engineers have different methods of reducing the concentration of the stresses in this points.

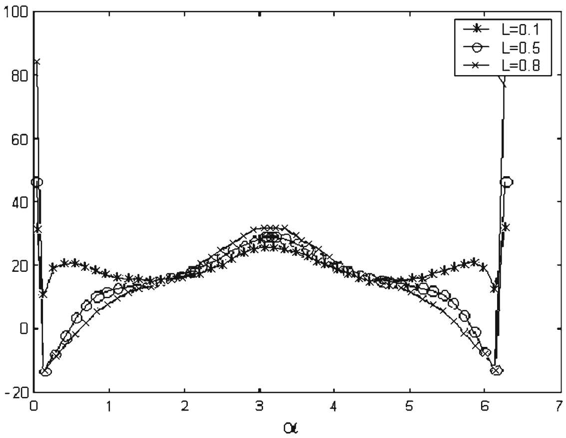

Fig. 2 $\sigma_{\alpha\alpha}$ on the boundary of an elliptic hole with one crack

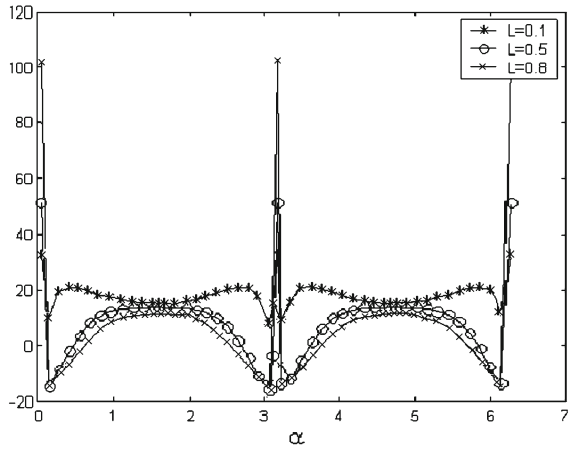

Fig. 3 $\sigma_{\alpha\alpha}$ on the boundary of an elliptic hole with two cracks

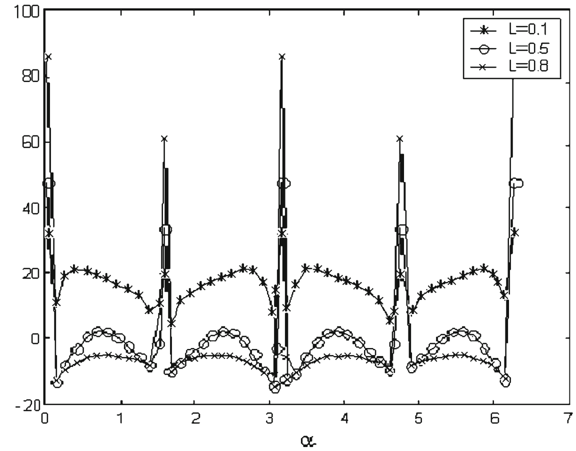

Fig. 4 $\sigma_{\alpha\alpha}$ on the boundary of an elliptic hole with four cracks

In the next section we will define boundary-value problems of elastic equilibrium for an elliptic ring with a cut.

3 Solution of boundary-value problems for an elliptic ring with a cut by the BEM

Below we consider two boundary-value problems for an elliptic ring with a cut along $\alpha = 0$ (Problem 1) in one case, and along $\alpha = \frac{\pi}{2}$ (Problem 2) in the other.

Problem 1 We formulate the problem as follows: in the domain $\Omega_1 = \{\theta_1 < \theta < \theta_2,\ 0 < \alpha < \pi\}$ (Fig. 5a), find the solution of the system of equilibrium equations (2) with the following boundary conditions:

$$\begin{aligned}
&\text{(a) when } \theta = \theta_1\text{: } \sigma_{\theta\theta} = p\cos\frac{\alpha}{2}, \quad \sigma_{\theta\alpha} = 0,\\
&\text{(b) when } \theta = \theta_2\text{: } \sigma_{\theta\theta} = 0, \quad \sigma_{\theta\alpha} = 0,\\
&\text{(c) when } \alpha = 0\text{: } v = 0, \quad \sigma_{\theta\alpha} = 0,\\
&\text{(d) when } \alpha = \pi\text{: } u = 0, \quad \sigma_{\alpha\alpha} = 0.
\end{aligned} \tag{13}$$

This problem coincides with the boundary-value problem for the elastic equilibrium of an elliptic ring with a cut (Fig. 1d) on whose contours the symmetry conditions (13c) are fulfilled.

Problem 2 In the domain $\Omega_2 = \{\theta_1 < \theta < \theta_2,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\}$ (Fig. 5b), find a solution of the system of equilibrium equations (2) with the following boundary conditions:

$$\begin{aligned}
&\text{(a) when } \theta = \theta_1\text{: } \sigma_{\theta\theta} = p\cos\left[\frac{1}{2}\left(\alpha - \frac{\pi}{2}\right)\right], \quad \sigma_{\theta\alpha} = 0,\\
&\text{(b) when } \theta = \theta_2\text{: } \sigma_{\theta\theta} = 0, \quad \sigma_{\theta\alpha} = 0,\\
&\text{(c) when } \alpha = -\frac{\pi}{2}\text{: } u = 0, \quad \sigma_{\alpha\alpha} = 0,\\
&\text{(d) when } \alpha = \frac{\pi}{2}\text{: } v = 0, \quad \sigma_{\theta\alpha} = 0.
\end{aligned} \tag{14}$$

This problem coincides with the boundary-value problem describing the elastic equilibrium of an elliptic ring with a cut (Fig. 1e) on whose contours the symmetry conditions (14d) are fulfilled.

The above-formulated problems (2), (13) and (2), (14) are solved by the boundary-element method. At the characteristic points of the domain we calculate the stress values for $\nu = 0.3$, $E = 7 \times 10^4$ (N/m^2), $\theta_1 = 0.01$ and 1 (m), $\theta_2 = 100$ (m), $p = 10$ (N/m^2). The semi-ellipses $\theta = \theta_1$ and $\theta = \theta_2$ are divided into 180 equal arcs, and the linear parts of the boundary are divided into 40 equal segments.

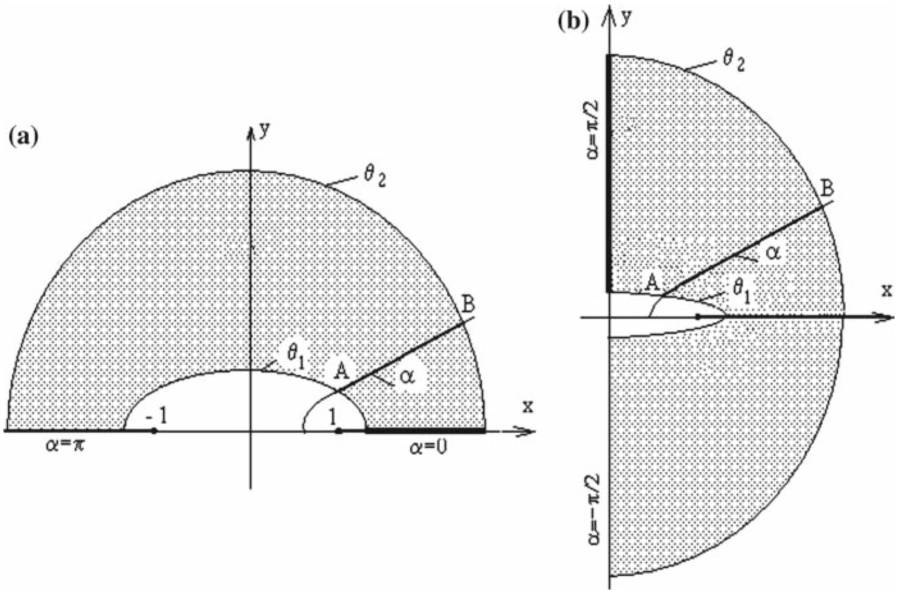

Fig. 5 Semi-elliptic rings

In the next section we will solve the problems analytically for an infinite domain with an elliptic hole and compare the results with the numerical solution obtained for $\theta_2 \gg \theta_1$.

4 Solution of the boundary-value problems for an infinite domain with an elliptic hole and a crack by the method of separation of variables (MSV)

The MSV yields an analytical (exact) solution of the boundary-value problems (2), (13) and (2), (14) for $\theta_2 \to \infty$ (Fig. 5, cases a, b). The solution is constructed using its general representation by two harmonic functions φ_1, φ_2. The components of the displacement vector have the form

$$\begin{aligned}\overline{u} = &-\left(\sinh^2\theta_1 \coth\theta \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} + \frac{æ-1}{2}\frac{\partial\varphi_1}{\partial\alpha} - \frac{\partial\varphi_2}{\partial\theta}\right)\sinh\theta\cos\alpha \\ &-\left(\cosh^2\theta_1 \tanh\theta \frac{\partial^2\varphi_1}{\partial\alpha^2} + \frac{æ-1}{2}\frac{\partial\varphi_1}{\partial\theta} - \frac{\partial\varphi_2}{\partial\alpha}\right)\cosh\theta\sin\alpha,\end{aligned}$$

$$\begin{aligned}\overline{v} = &\left(\cosh^2\theta_1 \tanh\theta \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} + \frac{æ-1}{2}\frac{\partial\varphi_1}{\partial\alpha} - \frac{\partial\varphi_2}{\partial\theta}\right)\sinh\theta\cos\alpha \\ &-\left(\sinh^2\theta_1 \coth\theta \frac{\partial^2\varphi_1}{\partial\alpha^2} + \frac{æ-1}{2}\frac{\partial\varphi_1}{\partial\theta} - \frac{\partial\varphi_2}{\partial\alpha}\right)\cosh\theta\sin\alpha,\end{aligned}$$

and the stress-tensor components are written as

$$\begin{aligned}\frac{h^2}{\mu}\sigma_{\theta\theta} = &\left(2\sinh^2\theta_1 \coth\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\sinh\theta\cos\alpha \\ &-\left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2} - \frac{\partial^2\varphi_1}{\partial\alpha^2} - 2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\cosh\theta\sin\alpha \\ &-\frac{2\left[\cosh(2\theta_1) - \cosh(2\theta)\right]}{\cosh(2\theta) - \cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\cosh\theta\sin\alpha - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\sinh\theta\cos\alpha\right),\end{aligned}$$

$$
\begin{aligned}
\frac{h^2}{\mu}\sigma_{\alpha\alpha} = & \left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2} + 3\frac{\partial^2\varphi_1}{\partial\alpha^2} - 2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\cosh\theta\sin\alpha \\
& - \left(2\sinh^2\theta_1 \coth\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} + 3\frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\sinh\theta\cos\alpha \\
& + \frac{2\left[\cosh(2\theta_1) - \cosh(2\theta)\right]}{\cosh(2\theta) - \cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\cosh\theta\sin\alpha - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\sinh\theta\cos\alpha\right), \\
\frac{h^2}{\mu}\sigma_{\theta\alpha} = & -\left(2\cosh^2\theta_1 \tanh\theta \frac{\partial^3\varphi_1}{\partial\alpha^3} - \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha} - 2\frac{\partial^2\varphi_2}{\partial\alpha^2}\right)\cosh\theta\sin\alpha \\
& - \left(2\sinh^2\theta_1 \coth\theta \frac{\partial^3\varphi_1}{\partial\theta\partial\alpha^2} - \frac{\partial^2\varphi_1}{\partial\alpha^2} - 2\frac{\partial^2\varphi_2}{\partial\theta\partial\alpha}\right)\sinh\theta\cos\alpha \\
& + \frac{2\left[\cosh(2\theta_1) - \cosh(2\theta)\right]}{\cosh(2\theta) - \cos(2\alpha)}\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\sinh\theta\cos\alpha + \frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\cosh\theta\sin\alpha\right).
\end{aligned}
$$

The boundary conditions (13c, d) are satisfied if

$$
\varphi_1 = \sum_{j=1}^{2} A_{1j}\mathrm{e}^{-k(\theta-\theta_1)}\sin(k\alpha), \quad \varphi_2 = \sum_{j=1}^{2} A_{2j}\mathrm{e}^{-k(\theta-\theta_1)}\cos(k\alpha), \quad k = \frac{2j-1}{2}. \tag{15}
$$

The constants A_{ij} $(i, j = 1, 2)$ are defined if the boundary conditions (13a, b) are satisfied.

Let us now consider the process of obtaining the values of the coefficients A_{ij}. We rewrite (13a) in the equivalent form

$$
\begin{aligned}
& -\sinh(2\theta_1)\left(\frac{\partial^2\varphi_1}{\partial\theta\partial\alpha}\right)_{\theta=\theta_1} + \left(\frac{\partial\varphi_1}{\partial\alpha}\right)_{\theta=\theta_1} + 2\left(\frac{\partial\varphi_2}{\partial\theta}\right)_{\theta=\theta_1} = G_1(\alpha), \\
& -\sinh(2\theta_1)\left(\frac{\partial^2\varphi_1}{\partial\alpha^2}\right)_{\theta=\theta_1} + \left(\frac{\partial\varphi_1}{\partial\theta}\right)_{\theta=\theta_1} + 2\left(\frac{\partial\varphi_2}{\partial\alpha}\right)_{\theta=\theta_1} = -G_2(\alpha),
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
G_1(\alpha) &= \int (\cosh\theta_1 \sin\alpha\sigma_{\theta\theta} + \sinh\theta_1 \cos\alpha\sigma_{\theta\alpha})\, d\alpha \\
&= -\frac{p}{3}\cosh\theta_1\cos\left(\frac{3}{2}\alpha\right) - p\cosh\theta_1\cos\left(\frac{\alpha}{2}\right) + C\frac{p}{2}\cosh\theta_1, \\
G_2(\alpha) &= \int (\sinh\theta_1 \cos\alpha\sigma_{\theta\theta} - \cosh\theta_1 \sin\alpha\sigma_{\theta\alpha})\, d\alpha \\
&= \frac{p}{3}\sinh\theta_1\sin\left(\frac{3}{2}\alpha\right) + p\sinh\theta_1\sin\left(\frac{\alpha}{2}\right) + C\frac{p}{2}\sinh\theta_1,
\end{aligned}
$$

where C is integration constant. We substitute (15) in (16) and compare the coefficients of the same trigonometric functions in the resulting equalities. Thus we obtain the following system of linear algebraic equations:

$$
\begin{cases}
\frac{1}{2}\left[\frac{1}{2}\sinh(2\theta_1) + 1\right]A_{11} - A_{21} = -p\cosh\theta_1, \\
\frac{1}{2}\left[\frac{1}{2}\sinh(2\theta_1) - 1\right]A_{11} - A_{21} = -p\sinh\theta_1, \\
\frac{3}{2}\left[\frac{3}{2}\sinh(2\theta_1) + 1\right]A_{12} - 3A_{22} = -\frac{p}{3}\cosh\theta_1, \\
\frac{3}{2}\left[\frac{3}{2}\sinh(2\theta_1) - 1\right]A_{12} - 3A_{22} = -\frac{p}{3}\sinh\theta_1.
\end{cases}
$$

from which we have

$$
A_{11} = -p\mathrm{e}^{-\theta_1}, \quad A_{12} = -\frac{p}{9}\mathrm{e}^{-\theta_1}, \quad A_{21} = \frac{p}{8}\left(3\mathrm{e}^{\theta_1} + \mathrm{e}^{-3\theta_1}\right), \quad A_{22} = \frac{p}{72}\left(\mathrm{e}^{\theta_1} + 3\mathrm{e}^{-3\theta_1}\right).
$$

The boundary conditions (14c, d) are satisfied if

$$\varphi_1 = \sum_{j=1}^{2} C_{1j} \mathrm{e}^{-k(\theta-\theta_1)} \sin\left[k\left(\alpha + \frac{\pi}{2}\right)\right],$$

$$\varphi_2 = \sum_{j=1}^{2} C_{2j} \mathrm{e}^{-k(\theta-\theta_1)} \cos\left[k\left(\alpha + \frac{\pi}{2}\right)\right], \quad k = \frac{2j-1}{2}.$$

The coefficients C_{ij} $(i, j = 1, 2)$ are defined in the same way as the coefficients A_{ij} in the preceding case:

$$C_{11} = -p\mathrm{e}^{-\theta_1}, \quad C_{12} = \frac{p}{9}\mathrm{e}^{-\theta_1}, \quad C_{21} = \frac{p}{8}\left(3\mathrm{e}^{\theta_1} + \mathrm{e}^{-3\theta_1}\right), \quad C_{22} = -\frac{p}{72}\left(\mathrm{e}^{\theta_1} + 3\mathrm{e}^{-3\theta_1}\right).$$

It is important to note that the solution found in the domain Ω_1 (or Ω_2) can be continuously extended across the boundary $\alpha = \pi$ (or $\alpha = -\frac{\pi}{2}$) . As a result, we obtain the domain $\Omega_3 = \{\theta_1 < \theta < \theta_2 \to \infty,\ 0 < \alpha < 2\pi\}$ $\left(\text{or } \Omega_4 = \left\{\theta_1 < \theta < \theta_2 \to \infty,\ -\frac{3\pi}{2} < \alpha < \frac{\pi}{2}\right\}\right)$ with a cut. On the cut contours $\alpha = 0$ and $\alpha = 2\pi$ $\left(\text{or } \alpha = -\frac{3\pi}{2}\right.$ $\left.\text{and } \alpha = \frac{\pi}{2}\right)$ the conditions $v = 0$, $\sigma_{\theta\alpha} = 0$ are fulfilled. Therefore we have obtained an approximate and an exact solution in the annular domain with a cut on whose contours the symmetry conditions are fulfilled.

A comparison of the results obtained by the boundary-element method with those of the exact solution shows an excellent agreement (see Figs. 6–11). We can therefore conclude that the application of the BEM has proved to be highly feasible for solving the boundary-value problems considered in this paper.

By solving the problems corresponding to Figs. 6–11 we obtain an idea of the distribution of the internal stresses throughout the body. In particular, using the BEM and the MSV we have calculated the distribution of the stresses $\sigma_{\alpha\alpha}$, $\sigma_{\theta\alpha}$, $\sigma_{\theta\theta}$ along the line AB (Fig. 5, cases a, b) when $\alpha = \frac{\pi}{3}$. As seen from Figs. 6 and 9, the concentration of $\sigma_{\alpha\alpha}$ takes place on the contour $\theta = \theta_1$, while for $\theta \to \infty$, we have $\sigma_{\alpha\alpha} \to 0$, as should have been expected. Figures 7 and 10 show the tangential stress $\sigma_{\theta\alpha}$ whose value on the contour is equal to zero (i.e., the boundary condition is fulfilled). Near the contour within the body, $\sigma_{\theta\alpha}$ has a high value but as it moves away from the boundary, $\sigma_{\theta\alpha}$ gradually tends to zero, $\sigma_{\theta\alpha} \underset{\theta\to\infty}{\to} 0$ (the boundary condition is also fulfilled). Figures 8 and 11 show the normal stress $\sigma_{\theta\theta}$, which on the hole contour has a value equal to that resulting from the corresponding boundary condition. Further it decreases monotonically and gradually vanishes as required by the condition at infinity.

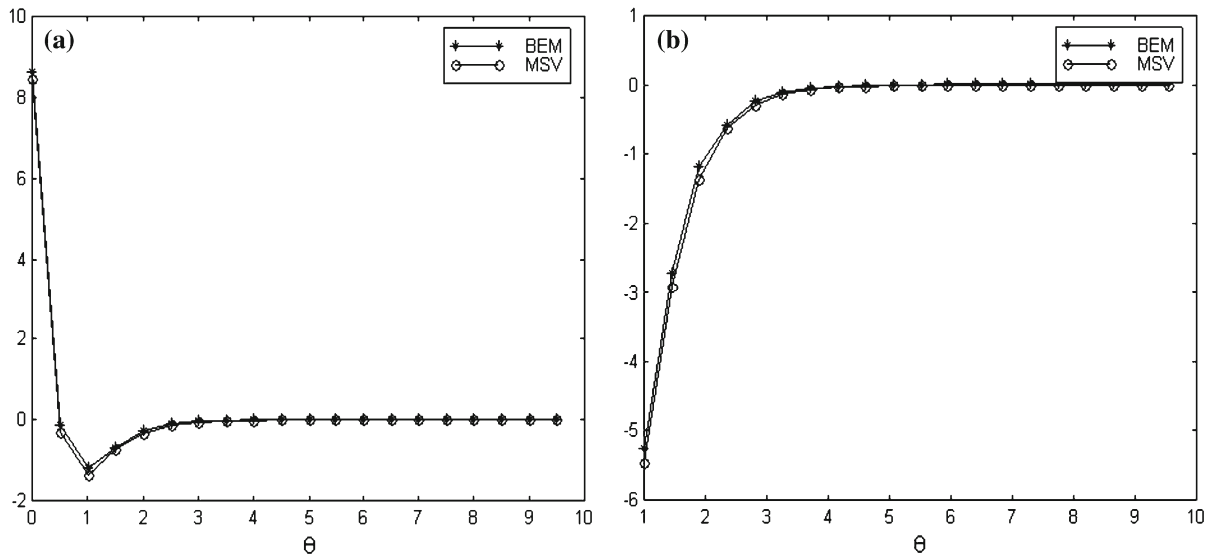

Fig. 6 The values of $\sigma_{\alpha\alpha}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \leq \theta < c_2$ in the case of a cut along $\alpha = 0$, where (**a**) $\theta_1 = 0.01$, $c_1 = 0.01$, $c_2 = 10$ and (**b**) $\theta_1 = 1$, $c_1 = 1$, $c_2 = 10$

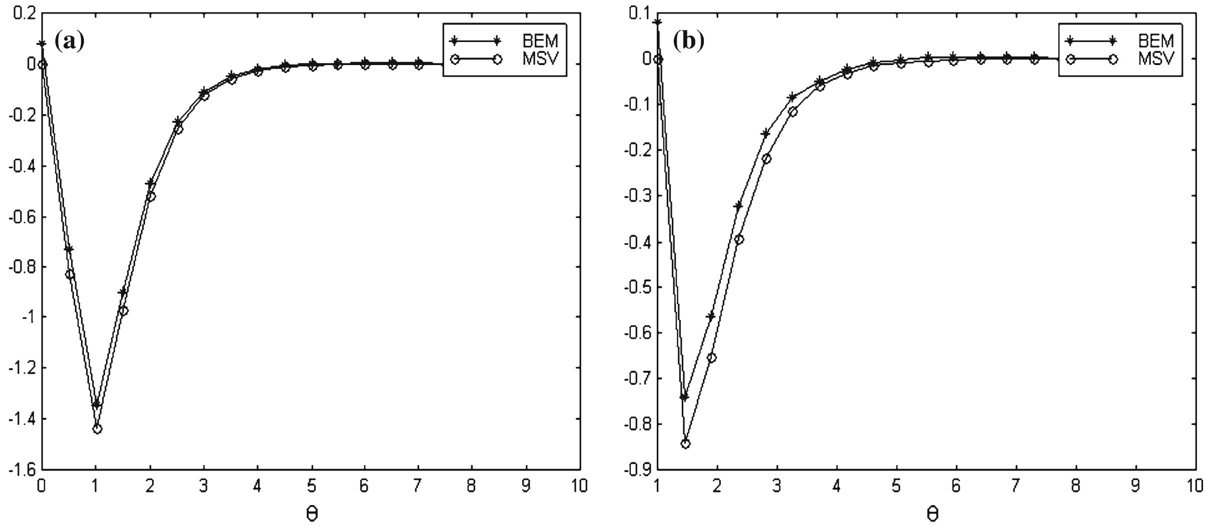

Fig. 7 The values of $\sigma_{\theta\alpha}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \le \theta < c_2$ in the case of a cut along $\alpha = 0$, where (**a**) $\theta_1 = 0.01$, $c_1 = 0.01$, $c_2 = 10$ and (**b**) $\theta_1 = 1$, $c_1 = 1$, $c_2 = 10$

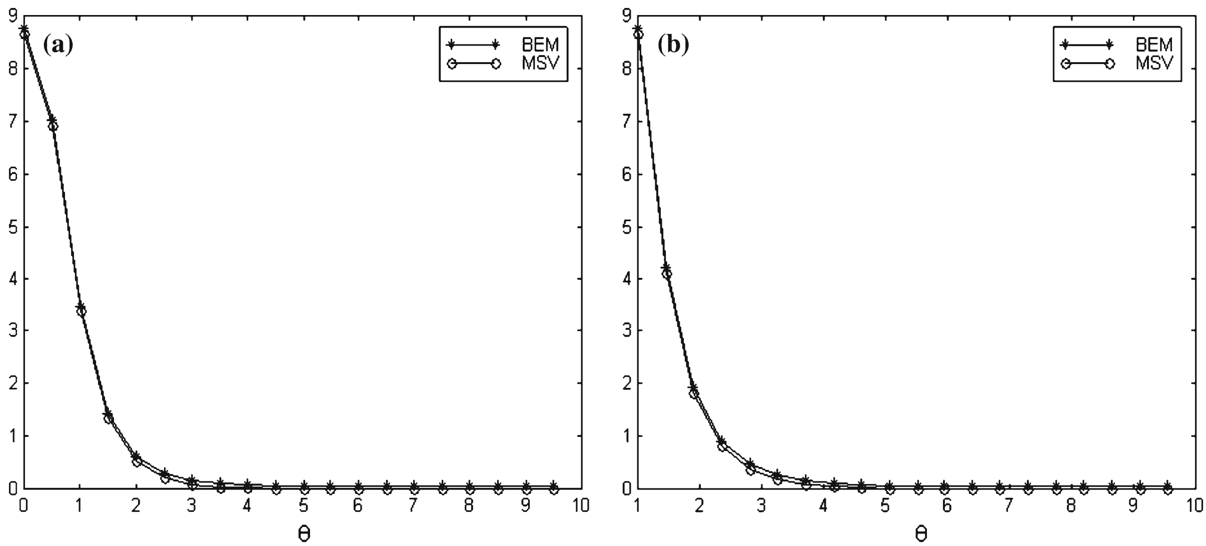

Fig. 8 The values of $\sigma_{\theta\theta}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \le \theta < c_2$ in the case of a cut along $\alpha = 0$, where (**a**) $\theta_1 = 0.01$, $c_1 = 0.01$, $c_2 = 10$ and (**b**) $\theta_1 = 1$, $c_1 = 1$, $c_2 = 10$

Table 1 contains both the approximate values (obtained by the BEM) and the exact values (obtained by the MSV) of solutions of Problem 1 for the stresses $\sigma_{\theta\theta}$, $\sigma_{\alpha\alpha}$ and $\sigma_{\theta\alpha}$ at some points from $\alpha = \frac{\pi}{3}$, $1 \le \theta < 10$, when the cut lies along $\alpha = 0$.

Table 2 contains the approximate values (obtained by the BEM) and the exact values (obtained by the MSV) of solutions of Problem 2 for the stresses $\sigma_{\theta\theta}$, $\sigma_{\alpha\alpha}$ and $\sigma_{\theta\alpha}$ at some points from $\alpha = \frac{\pi}{3}$, $0.1 \le \theta < 5$, when the cut lies along $\alpha = \frac{\pi}{2}$.

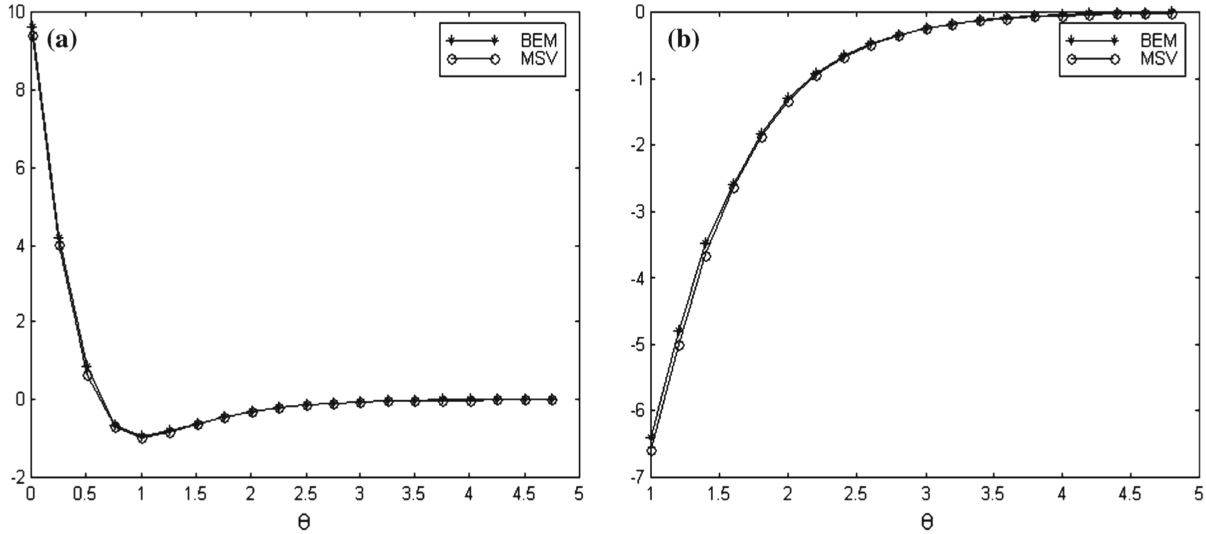

Fig. 9 The values of $\sigma_{\alpha\alpha}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \leq \theta < c_2$ in the case of a cut along $\alpha = \frac{\pi}{2}$, where (**a**) $\theta_1 = 0.01, c_1 = 0.01, c_2 = 5$ and (**b**) $\theta_1 = 1, c_1 = 1, c_2 = 5$

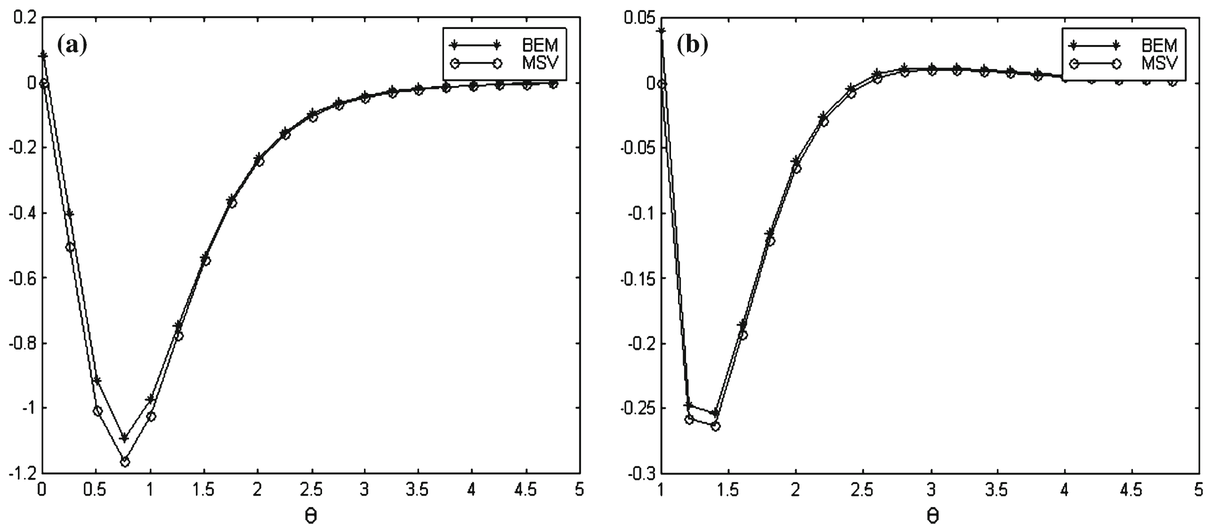

Fig. 10 The values of $\sigma_{\theta\alpha}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \leq \theta < c_2$ in the case of a cut along $\alpha = \frac{\pi}{2}$, where (**a**) $\theta_1 = 0.01, c_1 = 0.01, c_2 = 5$ and (**b**) $\theta_1 = 1, c_1 = 1, c_2 = 5$

5 Conclusions

The main results of this work can be formulated as follows:

1. The equilibrium equations (2) are written in terms of elliptic coordinates.
2. The solution of the equilibrium equation (2) is obtained by the method of separation of variables. The solution is constructed using its general representation by two harmonic functions.
3. Analytic (exact) solutions are obtained for two-dimensional boundary-value problems for the elastic equilibrium of infinite homogeneous isotropic bodies with elliptic holes and a cut on whose boundaries the symmetry conditions are fulfilled.

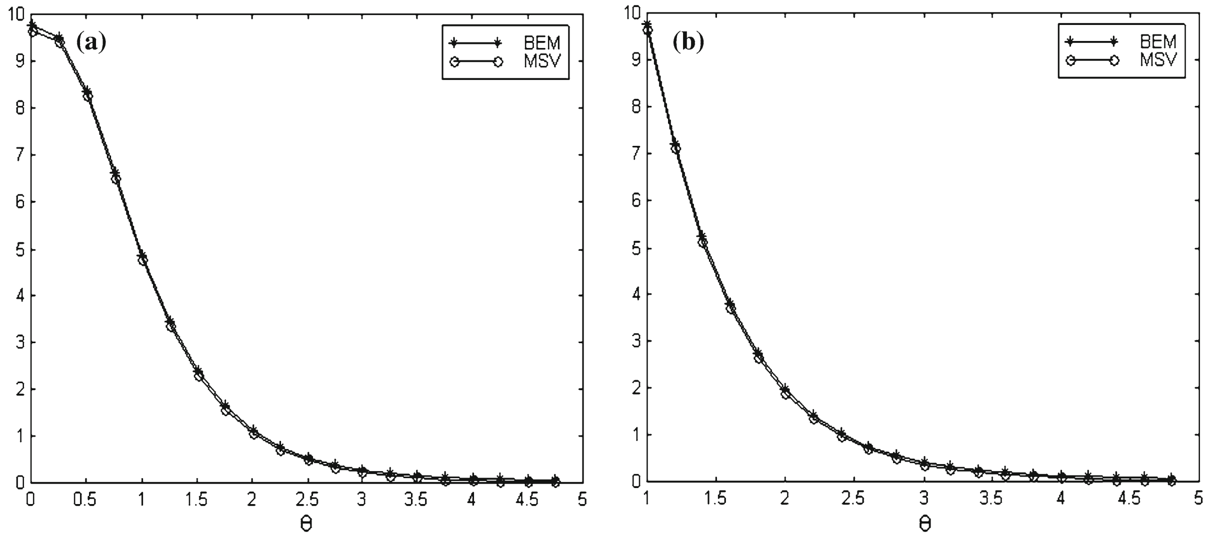

Fig. 11 The values of $\sigma_{\theta\theta}$ for $\alpha = \frac{\pi}{3}$ and $c_1 \le \theta < c_2$ in the case of a cut along $\alpha = \frac{\pi}{2}$, where (**a**) $\theta_1 = 0.01$, $c_1 = 0.01$, $c_2 = 5$ and (**b**) $\theta_1 = 1$, $c_1 = 1$, $c_2 = 5$

Table 1 Stresses at some points from $\alpha = \frac{\pi}{3}$, $1 \le \theta < 10$, when the cut lies along $\alpha = 0$

θ	$\sigma_{\theta\theta}$ Approxim.	Exact	$\sigma_{\alpha\alpha}$ Approxim.	Exact	$\sigma_{\theta\alpha}$ Approxim.	Exact
1.00	8.6961	8.6603	−5.4552	−5.4662	-1.0000×10^{-10}	0.0000
1.90	1.8268	1.8188	−1.3588	−1.3561	-6.5238×10^{-1}	-6.5441×10^{-1}
2.80	3.7391×10^{-1}	3.7091×10^{-1}	-2.8760×10^{-1}	-2.8980×10^{-1}	-2.1615×10^{-1}	-2.1535×10^{-1}
3.70	8.4566×10^{-2}	8.4385×10^{-2}	-6.5901×10^{-2}	-6.7401×10^{-2}	-5.9146×10^{-2}	-5.9486×10^{-2}
4.60	2.0907×10^{-2}	2.0609×10^{-2}	-1.6110×10^{-2}	-1.6630×10^{-2}	-1.5251×10^{-2}	-1.5761×10^{-2}
5.50	5.2986×10^{-3}	5.2086×10^{-3}	-4.1817×10^{-3}	-4.2217×10^{-3}	-4.0298×10^{-3}	-4.1198×10^{-3}
6.40	1.4161×10^{-3}	1.3361×10^{-3}	-1.0550×10^{-3}	-1.0850×10^{-3}	-1.0325×10^{-3}	-1.0715×10^{-3}
7.30	2.4468×10^{-4}	3.4488×10^{-4}	-3.7926×10^{-4}	-2.8026×10^{-4}	-2.8815×10^{-4}	-2.7815×10^{-4}
8.20	9.5249×10^{-5}	8.9249×10^{-5}	-7.2050×10^{-5}	-7.2550×10^{-5}	-3.2135×10^{-5}	-7.2145×10^{-5}
9.55	1.1860×10^{-5}	1.1770×10^{-5}	-9.3651×10^{-6}	-9.5691×10^{-6}	-7.5273×10^{-6}	-9.5253×10^{-6}

Table 2 Stresses at some points from $\alpha = \frac{\pi}{3}$, $0.1 \le \theta < 5$, when the cut lies along $\alpha = \frac{\pi}{2}$

θ	$\sigma_{\theta\theta}$ Approxim.	Exact	$\sigma_{\alpha\alpha}$ Approxim.	Exact	$\sigma_{\theta\alpha}$ Approxim.	Exact
1.0000×10^{-2}	9.6689	9.6593	9.4010	9.4039	-1.3345×10^{-10}	-2.3115×10^{-10}
5.0900×10^{-1}	8.2719	8.2709	6.5322×10^{-1}	6.5342×10^{-1}	−1.0043	−1.0067
1.0080	4.7699	4.7694	-9.8486×10^{-1}	-9.8500×10^{-1}	−1.0201	−1.0241
1.5070	2.2997	2.2961	-6.4044×10^{-1}	-6.4076×10^{-1}	-5.4406×10^{-1}	-5.4433×10^{-1}
2.0060	1.0596	1.0584	-3.0805×10^{-1}	-3.0812×10^{-1}	-2.4080×10^{-1}	-2.4083×10^{-1}
2.5050	4.8859×10^{-1}	4.8659×10^{-1}	-1.3395×10^{-1}	-1.3976×10^{-1}	-1.0233×10^{-1}	-1.0291×10^{-1}
3.0040	2.8318×10^{-1}	2.2518×10^{-1}	-6.3140×10^{-2}	-6.3210×10^{-2}	-4.4517×10^{-2}	-4.4520×10^{-2}
3.5030	1.0356×10^{-1}	1.0493×10^{-1}	-2.8767×10^{-2}	-2.8877×10^{-2}	-1.9123×10^{-2}	-1.9708×10^{-2}
4.0020	4.9189×10^{-2}	4.9153×10^{-2}	-1.3255×10^{-2}	-1.3335×10^{-2}	-8.8173×10^{-3}	-8.9093×10^{-3}
4.7505	1.5833×10^{-2}	1.5860×10^{-2}	-4.2462×10^{-3}	-4.2470×10^{-3}	-2.7549×10^{-3}	-2.7849×10^{-3}

4. Numerical solutions are obtained by the boundary-element method for two-dimensional boundary-value problems for the elastic equilibrium of infinite and finite homogeneous isotropic bodies with elliptic holes having cuts or cracks of finite length.
5. The question is investigated as to how the tangential stress concentration on the elliptic hole contour (except the crack ends) can be diminished by varying the number of cracks and their lengths.

The results shown in Figs. 2–4 lead to the following conclusions.

Strange as it might seem, an increase in the number of cracks and their lengths brings about a decrease of the stress concentration $\sigma_{\alpha\alpha}$ on the hole contour, except at the crack ends. Being aware of this fact, engineers create so-called technical cracks in order to fortify the tunnel structure. As for the crack ends, they apply various techniques to lower the stress concentration at these points.

The aim of the problems corresponding to Figs. 6–11 is to obtain a picture of stress distribution inside the body. In particular, the distributions of stresses $\sigma_{\alpha\alpha}$, $\sigma_{\theta\alpha}$, $\sigma_{\theta\theta}$ are calculated along the line AB (Fig. 5) for $\alpha = \frac{\pi}{3}$.

In Figs. 6–11 and Tables 1, 2 the calculated stress values are given for the points lying along $\alpha = \frac{\pi}{3}$ only for the sake of simplicity. Absolutely in the same manner we can define values for other points of the rings, provided they will not lie "too near" the boundary.

A comparison of the results obtained by the boundary-element method with those of the exact solution shows good agreement. Therefore, we may conclude that the application of the BEM has proved to be effective for the solution of the boundary-value problems considered in this paper, as well as of other problems involving a circular ring.

In conclusion, it should be said that we have obtained solutions of crack problems analogous to the ones considered here in the case of circular holes in elastic media consisting of binary mixtures. However, this will be the subject of a forthcoming paper.

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